the radius which perpendicularly bisects the director chord; even as it is also its intersection by the semi-diameter of ellipse or hyperbola, respectively bisecting the focal chords.

Corollary 3. If, then, with this point T as center, and radius Tf or TS, a circle be described [Figure 2], it will ever circumscribe the simple quadrangle $fESF$, formed by joining the respective foci f and S to the ends of the director chord.

While, on the other hand, it will ever be circumscribed by the quadrilateral $pPQq$; formed by joining the extremities of the two focal chords; and tangential to it at points $fESF$, where the major axis, and director chord, respectively, cut the sides pq, pP, PQ, and qQ; which points evidently coincide with the four corners of the inscribed quadrangle.

And furthermore, this circle will be either inscribed, or escribed to the said quadrilateral $pPQq$, according as PQ, the focal chord of the hyperbola, cuts branch S alone, or both the branches S and S'; thus causing the quadrilateral to be either simple—*e. g.* $pPQq$ [Figure 2]—; or re-entrant—as *e. g.* $p'P'Q'q'$ [Figure 3].

If, however, the chosen focal chords be through S'; with resultant double points T and T', then no circle can be drawn, since all the eight points, corresponding to $pqPQEFf$ and S, now lie collinearly upon the common determining diameter; which is also both focal and director chords.

Theorem 11. If h, upon the major axi, be the point of intersection by the director chord EF, and thus a summit of the quadrangle $fESF$ inscribed to the circle, whose radius is Tf or TS; it will also be the point of intersection of two diagonals pQ and qP of the quadrilateral $pPQq$, circumscribing that same circle, and tangential to it at points $fESF$ [Figure 2].

For the four points $fESF$, being thus common to all three figures—*i. e.* the circle about T, its inscribed quadrangle, and its circumscribing quadrilateral —they must therefore possess in common the self-conjugate triangle hJK.

Corollary 1. Thus J and K are also points in common, being both the other two summits of the inscribed quadrangle, and also the other two intersections of diagonals PqJ, pQK, and $S'JK$, to the circumscribing quadrilateral $pPQq$. So that lines Ef, SF, and Pq are ever concurrent; as also ES, fF, and pQ; and concurrent, too, on a common radius $S'JK$.

Theorem 12. Therefore, even as the fixed points f and S of the two reciprocal curves harmonically divide the diameter $[2S'D]$ of their common director circle; so also do any two corresponding points upon the curves harmonically divide the common radius or diameter determining them.

For, taking the circumscribing quadrilateral $pPQq$ [Figure 2], and considering it also as a quadrangle, then point h, where the director chord cuts the major axi [Theorem 11] will not only be the internal intersection of the diagonals pQ and qP; but also one of its three summits, of which the other two are S' and G. While similarly, taking the inscribed quadrangle $fESF$, and considering it also as a quadrilateral, since its three summits coincide with the three in-

tersections of the diagonals of $pPQq$ [Theorem 11, Corollary 1]; so conversely, must the intersections of its three diagonals—*i. e.* S', G, and h—coincide with the three summits of $pPQq$.

And thus the pencil ray of Gh must pass through F and E. So that $S'p:S'P=Ep:EP$; and $S'q:S'Q=Fq:FQ$, etc. While, since corresponding focal chords can be drawn through any two corresponding points—such as $p'q'$ and $P'Q'$; or $p'S'P'$—we can in like manner prove that $S'p':S'P'=Jp':JP'$ [Figure 3].

Theorem 13. If pm and PM be the perpendiculars from the reciprocal points p and P respectively, to their common directrix; and E be the extremity of their common determining radius $S'pEP$; then will lines EM and Em pass through f and S, the fixed points [Figure 2].

For $pE:pm=fp:pm=fa:aX=S'f:a'a=S'f:S'D=S'D:S'S=S'E:S'S$; while pm is parallel to $S'S$; and thus angles Epm and $ES'S$ are equal. Therefore Epm and $ES'S$ are similar triangles; and EmS one right line. And in a like manner, $PM:PE=PM:SP=AX:SA=A'A:S'S=S'D:S'S=S'f:S'D=S'f:S'E$; while angles MPE and $ES'f$ are equal. Therefore MEP and fES' are also similar triangles; angles MEP and fES' equal; and thus MEf one right line.

If Q' be taken on the other branch of the hyperbola, with $Q'S'q'F'$ for its determining diameter; and $Q'M'$, and say $q'm'$ for its perpendiculars to the directrix; then similarly FfM will be one right line; and also $Fm'S$.

Corollary 1. pEm and PME, like $S'ES$ and $S'fE$, are similar triangles; and likewise $q'F'm'$ and $Q'MF'$; $S'FS$ and $S'fF'$.

Theorem 14. The corresponding chords of reciprocal curves, and their common director chord, all three meet in one point upon their common directrix [Figures 2 and 3].

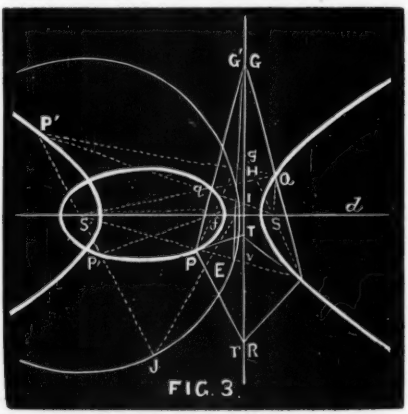


FIG. 3

First, let the reciprocal focal chords pq and PQ, and their director chord EF, determined by the two radii $S'pEP$ and $S'qFQ$, be the three chords in question [Figure 2], and let pq cut the directrix in, say g; and PQ cut it in G; we will first prove g and G to be identical.

For angles qfa, or gfX, and QSA, or GSX, are equal [Theorem 6]; while fX equals SX [Theorem 7]. So that the right angled triangles fXg and SXG are equal in all respects; Xg to XG, and thus g and G are the same point. Therefore the reciprocal focal chords pqG and PQG meet in the same point G upon the directrix.

Next, G being thus a summit; S' another summit; and $S'pEP$, $S'qFQ$, harmonic ranges upon $pPQq$ considered as a quadrangle [Theorem 12], it follows that GS', Gqp, GFE, and GQP form a harmonic pencil. So that the director chord EF meets its reciprocal focal chords pq and PQ also in G upon the directrix.

And similarly, if $p'fq'I$ and $P'IQ'$S be the two focal chords chosen [Figure 3], with their common director chord JK, determined by radius $S'q'KQ'$, and diameter $P'S'p'J$; for all three can in like manner be shown to meet in I upon the directrix. While if focal chords through S' be the ones selected, still more evidently do they, and their director chord, meet the directrix in one point; all three chords being coincident with each other, and with the determining diameter.

Secondly, let two non-focal reciprocal chords be chosen, such as pq and PQ [Figure 3]; or $p'q$ and $P'Q$. Then will they, and their common director chord, EF, or JF, as the case may be, still all three meet the directrix in one point G, or H. For, let chord pq cut the directrix in, say, G', and PQ cut it in G; then G and G' are identical. Join points pf, qf, PS, QS, $G'f$, and GS; and let the corresponding focal chords through p and P cut the directrix, as we have just seen, in one and the same point g.

Then, by a well known theorem, $G'f$ externally bisects the focal angle pfq of chord pqG'; and similarly, GS externally bisects angle PSQ. But these angles pfq and PSQ are equal, since $pfS'=PSd$; and $qfS'=QSd$ [Theorem 6]; and thus the halves of their supplements, $G'fg$ and GSg, are also equal, even as also are gfX and gSX. Therefore the right angled triangles $G'fX$ and GSX are equal in all respects; with $G'X$ equal to GX; and thus G' and G are the same point.

Taking $pPQq$ as a quadrangle, G is thus one of its three summits, S' another summit; and $S'pEP$, $S'qFQ$ harmonic ranges upon it; so that GS', Gqp, GFE, and GQP form a harmonic pencil, with ray GFE passing through the third summit; which is also the intersection of diagonals pQ and Pq. And therefore the two non-focal reciprocal chords pqG and PQG with their director chord EF, all three meet in G upon the directrix.

While, if $p'qH$ and $P'HQ$ be the two reciprocal non-focal chords chosen, with their director chord JFH [Figure 3]; in like manner it can be proved that they all three meet the directrix in H. For this is the meeting point of the external and internal bisectors, Hf and HS, of the focal angles $p'fq$ and $P'SQ$, respectively; since the focal chords through p' and P' meet, as we have seen, in I upon the directrix; and the focal angle $p'fq$ is equal to the supplement of focal angle $P'SQ$; $p'fS'$ being equal to $P'SS'$; and qfS' to QSd [Theorem 6], and thus the right angled triangles HXf and HXS are equal in all respects; and chords $p'qH$ and $P'HQ$ meet in the same point H upon the directrix. Which is also a summit of the re-entrant quadrangle $P'Qqp'$; another summit being S'; $S'qFQ$ and $P'S'p'J$ harmonic ranges upon it [Theorem 12]; and thus HS', Hqp', HFJ, and $P'HQ$ an harmonic pencil. So that the reciprocal non-focal chords $p'qH$ and $P'HQ$, with their common director chord JFH, all three meet in one and the same point H upon the common directrix. While the external intersection of the lines through $P'q$ and $p'Q$, since this is the third summit to the re-entrant quadrangle $P'Qqp'$, must lie on the director chord HFJ, since it is a pencil ray; even as the internal summit h [Theorems 11 and 12. Figure 2] lay upon the director chord and pencil ray EF.

Corollary 1. Conversely, then, chords through any two corresponding

points, such as pP, or $p'P'$ [Figure 3], determined by a common radius $S'pEP$, or diameter $P'S'p'J$; and meeting the directrix in the same point G, or H, etc., are thereby corresponding chords, with two other corresponding points qQ, etc. While EFG, or JFH, etc., will be their director chord, cutting the director circle again in F, etc., the extremity of the common radius $S'qFQ$, or diameter, determining these two new points.

Theorem 15. Of two such reciprocal chords, if one pq', or PP' passes through the center [O or C] of its curve, then the other PQ', or pp', as the case may be, is thereby parallel to the major axis, and thus meets its fellow pq', at M, or m, its perpendicular intersection with the directrix. While $F'EM$, or $E'Em$, the director chord in question, has become collinear with fE and fF', or SE and SE', the two lines joining the extremities of the determining radius and diameter to f or S, the fixed point of that chord which passes through the center [Figure 2].

For, let chord pq' of the ellipse pass through its center O; and chord PMQ' be drawn in the hyperbola parallel to the major axis, thus cutting the directrix perpendicularly in M; and being the directrix distances of P and Q'. Then, both ellipse and hyperbola being symmetrical about their respective major and minor axi; angles $q'fS'=pSf=PS'S=Q'SS'$; and thus $q'Q'$ are corresponding points, lying on the common determining diameter $Q'S'q'F'$ [Theorem 6]; while pq' and PQ' are reciprocal chords, with M for their meeting point on the directrix [Theorem 14].

And, in like manner, if $PmCP'$ be drawn through the center of the hyperbola, and $p'pm$ be parallel to the major axis of the ellipse, $P'SS'=PS'S=pSf=p'fS'$; so that pp' is the reciprocal chord to PP', and meets it at m on the directrix, the common foot of the directrix distances pm and $p'm$.

Next, the director chord $F'EM$ is collinear with both fE and fF'. For $pf=pE$, and $S'F'=S'E$; so that fpE and $F'S'E$ are isosceles triangles, with sides $S'E$ and pE collinear, and angles pfE, pEf, and $S'F'E$ equal. And thus their bases $F'E$ and fE are also collinear; and coincident with $F'EM$ the director chord [Theorem 13]. Similarly, if $PmCP'$ and mpp' be the two chords in question, with $E'EM$ for their director chord; SPE and $E'S'E$ are isosceles triangles, with sides PE and $S'E$ collinear; while $E'Em$, being Em produced, must pass through S [Theorem 13]. And thus their bases SE and $E'E$ are also collinear with each other, and with $E'Em$ this director chord.

Corollary 1. Therefore, in such a case, $pfqG$ is parallel to $Q'S'q'F'$, or $PSQG$ to $P'S'p'E'$.

Corollary 2. Obviously, if pP lie on a common radius $S'pEP$, then $q'Q'$ or $p'P'$ must lie on a diameter; and conversely.

Corollary 3. Points p' and q', or P' and Q', are also symmetric, their chords $p'q'$ and $P'Q'$ being perpendicular to the major axis. And thus $P'q'$ and $Q'p'$ are equally inclined to the axi; and likewise $P'S'p'E'$ and $Q'S'q'F'$.

Theorem 16. The normals of corresponding points meet upon the common directrix [Figure 3]. Let pP be the two corresponding points; pr and PR their

respective normals, cutting the directrix in say r and R; $SpEvP$ their determining radius, meeting the directrix in v; and finally pT and PT their respective tangents, meeting each other in T upon the directrix [Theorem 10]. Let $pfp''g$ and $PSP''g$ be the focal chords through p and P, meeting each other in g upon the directrix [Theorem 14], and determining the other two corresponding points p'' and P'' [Theorem 9].

Then, since pr and pT internally and externally bisect the focal angle Spf, and thus also vpg; therefore $rv:rg=Tv:Tg$ [Euclid VI, 3 and A]. And similarly, SPS, or vPg is internally and externally bisected by PT and PR; so that $Rv:Rg=Tv:Tg$. Therefore, since g, T, and v are identical points in both harmonic ranges, points r and R are also identical.

Theorem 17. The normals of the corresponding focal chords, pp'' and PP'' respectively, meet upon the radius passing through T, the common intersection of their tangents with the directrix, and bisecting their common director chord [Theorem 10, Corollary 2].

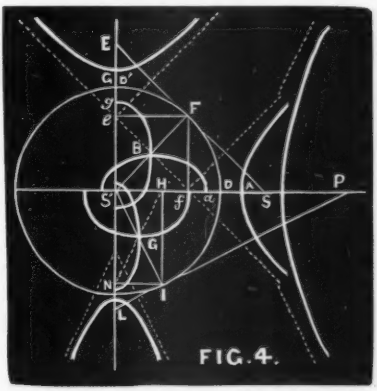


FIG. 4.

For, if such radius, say $Sr'n'TN'R'$ be drawn, meeting the focal chords pp'' and PP'' through p and P in n' and N' respectively; and the normals of p and P in r' and R'. Then, since pr', and pT, PT, and PR', internally and externally bisect the focal angles Spf and SPS, or Spn' and SPN'; therefore $ST:Tn'=r'S:r'n'$; and $ST:TN'=R'S:R'N'$. While, if the normals of p'' and P'' cut the radius $Sr'n'TN'R'$ in, say, r'' and R'' respectively; then similarly $ST:Tn'=r''S:r''n'$, and $ST:TN'=R''S:R''N'$. And S, T, n' and N' being identical points upon the four harmonic ranges $Sr'n'T$, $Sr''n'T$, $STN'R'$, and $STN'R''$; therefore r' and r'' are also identical points; and likewise R' and R''.

And thus the normals of pp'' meet in r'; and those of PP'' in R', upon the radius $Sr'n'TN'R'$ through T, the common intersection of their four tangents with the directrix; which radius also bisects the common director chord of the reciprocal focal chords.

The foregoing theorems will thus enable us to determine the *Reciprocal* and *Anacyclic* forms to any given conic curve. For the passage of our fixed point, from being the center of its director circle, to a position at infinity, external to said circle, as the focus of a degenerated right line hyperbola, this process, I say, can be now conceived as part of a simultaneous four-fold movement. Since at any point in this course—say, for example, at f [Figure 4]—its generated curve, in this case an ellipse, is balanced and answered by a reciprocal curve, in this case a hyperbola; reverse and opposite in every corresponding point, and gener-

ated by a second fixed point [*e. g. S*], harmonic to the first; rushing from the polar of the center at infinity to meet it on the circumference; and sinking ultimately into the center from whence the original fixed point arose, as that point passes at last to the infinity from whence sprang its fellow. The medial and pivotal conic form in this cycle, as previously stated, being the parabola.

But in addition to these two greater reciprocals, there are also generated two *anacyclic* curves, if we may so name them, or minor reciprocals, in the elliptic and hyperbolic cycles, respectively.

For just as the sine and tangent of the same point [*e. g. F*, or *I*, Figure 4] on our director circle gave us, on the major axis, the harmonic fixed points of a reciprocal ellipse and hyperbola; so also will the cosine and cotangent of this same point further determine, on the minor axis, two more harmonic fixed points, which will generate a second pair of reciprocal curves—ellipse and hyperbola—anacyclic, and opposite, in many ways, each to its fellow of like form in the first pair. In the hyperbola this anacyclic curve will be a conjugate hyperbola; conjugate, however, not with respect to its position, or magnitude; but solely in regards to its eccentric ratio, or shape. For the common radius [*e. g. S' F*, or *S' I*], whose tangent and cotangent pass through the two fixed points generating the said anacyclic hyperbolas, is thereby parallel to an asymptote of each [Theorem 4]; and the said asymptotes, being thus themselves parallel, and so inclusive of supplementary angles with their fellows, must belong to hyperbolas conjugate in ratio. Nevertheless, since the asymptotes are not coincident, but lie on opposite sides of the common determining radius, their respective hyperbolas are not conjugates with respect to position. Nor, again, are they with reference to their magnitude; except, indeed, they be equilaterals; since having a common director circle, their major axi are equal.

The hyperbolic anacyclics are thus conjugates in ratio. And although the same does not hold true of their reciprocal anacyclic ellipses, since the conjugate to an ellipse is never other than itself; yet nevertheless, as the reciprocals of conjugates, they will hold most important relations to each other.

Thus, for example, the right line *fe*, or *HK*, joining the anacyclic fixed points, will intersect their common determining radius *S'F*, or *S'I*, on a common extremity, *B*, or *b*, of the respective minor axi. Since, joining the fixed points, it is necessarily the bisecting and bisected diagonal to the parallelogram under the sine and cosine of *F*, or *I*; and its center *B*, or *b*, is thus also the center for the concyclic points *FfS'e*, or *IHS'K*, as the case may be; and therefore, again, is a common extremity of the minor axi [Theorem 3].

The interfocal distance *S'f*, or *S'K*, must therefore ever equal the minor axis of its anacyclic ellipse; since points *S'* and *B*, or *b*, are common; while if *C* and *C'*, or *C''* and *C'''*, represent the respective centers of the two anacyclic curves; $S'C=BC'$, and $S'C'=BC$, or $S'C''=bC'''$, and $S'C'''=bC''$.

And so again; if *X* be taken to represent the magnitude *S' F*, or *S' D*, the radius of the common director circle, equalling the major axis; and *Y* be taken to represent the interfocal magnitude *S' f* of the given ellipse; then the eccentric

ratios of the said ellipse, and of its reciprocal hyperbola will be represented by $Y:X$, and $X:Y$ respectively [Theorem 7]. While the eccentricity of the anacyclic hyperbola, as a conjugate in ratio, will obviously be $X:\sqrt{(X^2-Y^2)}$; and thus the ratio of its reciprocal ellipse—the anacyclic to the first ellipse—will be $\sqrt{(X^2-Y^2)}:X$.

As the reciprocal fixed points then, say P and H [Figure 4], of ellipse and hyperbola move from the center and infinity respectively, towards the circumference and each other, along the major axis, their anacyclic fixed points, L and K, are moving simultaneously in a reverse direction along the minor axis, away from the circumference and each other, towards the center or infinity. Instantaneously transposing with each other, and returning from thence back to the circumference; as the two parent points meet and cross on the circumference and pass onwards towards infinity and the center. The common determining radius—as *e. g. SI*—during these cycles, swinging from the minor to the major axis, and back again. The diagonal—as *e. g. HK*—and the tangent and cotangent—as *e. g. PIL*—meanwhile rocking from a position, on the minor axis, at right angles to each other, through the medial parallels, corresponding to *eBf* and *EFS*, to a second position at right angles on the major axis; and then, reversing, back to their original position.

The medial forms, therefore, of both the hyperbolic and the elliptic cycles —lying midway between the parabola, the pivotal form of the greater reciprocal cycle, on the one hand; and its extremes, the circle and right line hyperbola at infinity, on the other—are alike determined by that radius *SF*, which bisects in 45° the right angle between the axi. Giving us, then, outside of the director circle, two equilateral hyperbolas. And within it, two precisely similar ellipses, whose eccentric ratios, like those of their reciprocal equilateral hyperbolas, will be as $\sqrt{1}$ to $\sqrt{2}$, or $\sqrt{2}:\sqrt{1}$; so that the duplicate ratio of *CS* to *CX*, or *CX* to *CS*, in both alike, is as 1 to 2.

This medial ellipse, as we may call it, in default of a better term, has many important theorems in common with its reciprocal, the equilateral hyperbola; despite the fact that the elliptic, answering to the latter curve, is usually thought to be the circle. Yet a lack of space compels me to defer to a future occasion the discussion of some of these theorems; as well as others pertaining in general to "reciprocals" and "anacyclics."

USES OF THE SPECIAL TRIPLE PRODUCT $\mathbf{ab^2}$ OF EXTENSIVE QUANTITIES.

By J. V. COLLINS, Stevens Point, Wisconsin.

1. The study of the special triple product of extensive quantities where the second factor is repeated gives interesting results. Thus by means of this product the nature of the quaternion itself can be developed directly from the fundamental laws of multiplication. The author of the *Ausdehnungslehre* developed his subject by using the fundamental laws of operation as a starting point, but writers on quaternions pursue a widely different course.

2. Following out the idea of developing quaternions from the fundamental laws assumed to hold *for products of the reference units only*,* the writer used the special triple product $\mathbf{aB^2}$ where $\mathbf{a}$ and $\mathbf{B}$ are extensive quantities, $\mathbf{B}$ being a unit quantity, rather than the dual product $\mathbf{ab}$, or the more general triple product $\mathbf{abc}$. By Hamilton's law† $\mathbf{aB^2}=-\mathbf{a}$. Then $\mathbf{(aB)B}$ by the associative law should likewise equal $-\mathbf{a}$. Expressing both $\mathbf{a}$ and $\mathbf{B}$ as sums of components along three rectangular reference axes, we may write

$$\mathbf{a}=a_i\mathbf{i}+a_j\mathbf{j}+a_k\mathbf{k},$$

$$\mathbf{B}=B_i\mathbf{i}+B_j\mathbf{j}+B_k\mathbf{k}.$$

Then if M_i, M_j, M_k are the respective minors of m_i, m_j, m_k in the determinant whose respective rows are m_i, m_j, m_k, a_i, a_j, a_k, B_i, B_j, B_k,

$$\mathbf{(aB)B}=-\mathbf{a}=-(a_iB_i+a_jB_j+a_kB_k)\mathbf{B}+\begin{vmatrix} M_j & M_k \\ B_j & B_k \end{vmatrix}\mathbf{i}+\begin{vmatrix} M_k & M_i \\ B_k & B_i \end{vmatrix}\mathbf{j}+\begin{vmatrix} M_i & M_j \\ B_i & B_j \end{vmatrix}\mathbf{k}.$$

For, one has no trouble in interpreting by means of the assumed laws for the products of the units such terms as $\mathbf{(ij)i}$, $\mathbf{(ij)j}$, etc. Then only the three terms which contain the product of all three reference units remain. But after rearranging the terms which contain these products by the commutative and associative laws, and combining, the coefficient of the product vanishes identically. The simplified product just given shows $-\mathbf{a}$ "analyzed" into two components, one along $\mathbf{B}$ and the other perpendicular to it.

If then $\mathbf{(aB)B}=-\mathbf{a}$ we may assume

$$\frac{-\mathbf{a}}{\mathbf{B}}=\mathbf{aB},$$

*See Grassmann's Ausdehnungslehre of 1862, Engel ed., Arts. 37, 42.

†It is easy to prove that if the squares of the reference units equal negative unity, the square of any unit vector has the same value.

which evidently makes of $\mathbf{aB}$ a kind of multiplier or operator that can change $\mathbf{B}$ into $-\mathbf{a}$.

Let us now take the case of three unit quantities, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, of the same kind.

$\mathbf{AB}$ is an operator which changes $\mathbf{B}$ into $-\mathbf{A}$,
$\mathbf{BC}$ is an operator which changes $\mathbf{C}$ into $-\mathbf{B}$,
$\mathbf{CA}$ is an operator which changes $\mathbf{A}$ into $-\mathbf{C}$,
$\mathbf{BA}$ is an operator which changes $\mathbf{A}$ into $-\mathbf{B}$, etc.

Attempting to satisfy these conditions one is led directly to put $\mathbf{AB}=\mathbf{C}$, which by our assumed laws immediately gives

$$\mathbf{BC}=\mathbf{A},\quad \mathbf{CA}=\mathbf{B},\quad \mathbf{CB}=-\mathbf{A},\quad \mathbf{AC}=-\mathbf{B},\quad \mathbf{BA}=-\mathbf{C}.$$

These equations with those already assumed, viz., $\mathbf{A}^2=\mathbf{B}^2=\mathbf{C}^2=-1$, are the fundamental laws of quaternions. From them, as Tait states,* the whole subject can be developed. We have already seen that $\mathbf{aB}$ is an operator which changes $\mathbf{B}$ into $-\mathbf{a}$: an easy extension interprets the meaning of $(\mathbf{aB})\mathbf{c}$.

Thus starting from the fundamental laws of multiplication only, one is led naturally and directly to the fully developed subject of quaternions or to Macfarlane's *Algebra of Physics*. In this way unity in the form of presentation is secured for all branches of vector analysis. Moreover there is nothing in the preceding to limit the meaning to vectors. Thus $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{a}$, $\mathbf{b}$ may stand for points, lines, and planes and the results be interpreted. This method of developing the subject seems much preferable to the one followed by writers on quaternions who begin arbitrarily with either the "$\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$" system of vectors, or with the conception of a quaternion as an operator which changes one vestor into another, these assumptions or definitions being followed later by an investigation into the fundamental laws of multiplication. Moreover this plan has the additional merit of showing just why Hamilton made the square of a unit vector negative rather than positive unity. For had the scalar law chosen been $\mathbf{A}^2=\mathbf{B}^2=\mathbf{C}^2=1$, $(\mathbf{aB})\mathbf{B}$ would have come out $-\mathbf{a}$ instead of $\mathbf{a}$ as called for by the associative law. Thus the associative law has to be given up in the algebra of physics (unless the factor i is introduced) and one must write $(\mathbf{Ba})\mathbf{B}=\mathbf{a}$ instead of $(\mathbf{aB})\mathbf{B}=\mathbf{a}$. It should be added however that this does not deprive the algebra of physics of usefulness, for in some ways it is more practical than the quaternions it so closely resembles.

3. We have just seen to what use the product $\mathbf{aB}^2$ can be put when its parts are used together. Let us next study to what use they can be put when employed independently.

Let us denote extensive quantities by clarendon or black faced type, unit quantities by capitals, tensors by italics, and a perpendicular to a vector in positive direction of angle (in plane of factors) by writing a line over it. Then (§2) let

*Tait, Quaternions, Art. 71, 2nd ed.

$\mathbf{a}^{o}\,\mathbf{b}$=cosine part of product$-(\mathbf{ab})\mathbf{B}=ab.\cos\angle\mathbf{ab}.\mathbf{B}$,

$\mathbf{a}^{i}\,\mathbf{b}$=sine part of product$-(\mathbf{ab})\mathbf{B}=ab.\sin\angle\mathbf{ab}.\overline{B}$,

$\mathbf{a}^{t}\,\mathbf{b}$=tangent function of $\mathbf{a}$ and $\mathbf{b}=ab.\tan\angle\mathbf{ab}.\overline{B}$,

and $\mathbf{a}^{c}\,\mathbf{b}$=cotangent function, $\mathbf{a}^{z}\,\mathbf{b}$=secant function, and $\mathbf{a}^{k}\,\mathbf{b}$=cosecant function of vectors $\mathbf{a}$ and $\mathbf{b}$. Evidently $\mathbf{a}^{c}\,\mathbf{b}$, $\mathbf{a}^{z}\,\mathbf{b}$ differ only by a numerical factor from $\mathbf{a}^{o}\,\mathbf{b}$, both being measured in same direction as $\mathbf{a}^{o}\,\mathbf{b}$, while $\mathbf{a}^{t}\,\mathbf{b}$ and $\mathbf{a}^{k}\,\mathbf{b}$ differ in the same way from $\mathbf{a}^{i}\,\mathbf{b}$ and are measured in the same direction as $\mathbf{a}^{i}\,\mathbf{b}$.

4. By means of the expressions of the last articles one can write vector equations in a form corresponding to the cartesian ones, as $\mathbf{r}=\mathbf{x}+\mathbf{y}$, or $\mathbf{r}=\mathbf{x}+\mathbf{y}+\mathbf{z}$. Thus

$$\mathbf{r}=\mathbf{Q}^{o}\mathbf{a}+\mathbf{Q}^{o}\mathbf{b}$$

is the equation of an ellipse where $\mathbf{Q}$ is an auxiliary vector making with $\mathbf{a}$ the so-called eccentric angle, and $\mathbf{a}$ and $\mathbf{b}$ are the semi-axes. Evidently $\mathbf{Q}$ can be understood, whereupon the equation reduces to

$$\mathbf{r}=^{o}\mathbf{a}+^{o}\mathbf{b}.$$

Similarly $\mathbf{r}=^{z}\mathbf{a}-^{c}\mathbf{b}$ is the equation of the hyperbola referred to its axes, $\mathbf{r}=^{c^{2}}\mathbf{a}-^{c}\mathbf{a}$ that of the parabola, the dot under $\mathbf{c}$ denoting that the vector length is measured in the complementary direction to that of $\mathbf{c}$ unmarked, here along $\mathbf{b}$. The same notation is available for higher plane curves. Thus

$\mathbf{r}=^{z}\mathbf{a}\,^{z}\mathbf{b}$ corresponds to $x^{2}(y^{2}-b^{2})=a^{2}y^{2}$,

$\mathbf{r}=^{t}\mathbf{p}+^{o}\mathbf{m}+\mathbf{p}+^{i}\mathbf{m}$ corresponds to $m^{2}y^{2}=(p-y)^{2}(x^{2}+y^{2})$.

In space we have* $\mathbf{r}=^{o}\mathbf{a}+^{o}\mathbf{b}+^{o}\mathbf{c}$, as the vector equation of the ellipsoid, where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are vector axes, and the auxiliary vector $\mathbf{Q}$, understood, is not written. Similarly

$\mathbf{r}=^{z}\mathbf{A}(\mathbf{a}+^{o}\mathbf{b}+^{o}\mathbf{c})$ is the equation of the hyperboloid of two sheets,

$\mathbf{r}=^{k}\mathbf{C}[^{k}\mathbf{C}(^{o}\mathbf{a}+^{o}\mathbf{b})-^{o}\mathbf{c}]$ is the equation of the hyperboloid of one sheet,

$\mathbf{r}=^{c^{2}}\mathbf{c}+^{c}\mathbf{c}.^{z}\mathbf{a}(^{z}\mathbf{C}.\mathbf{a}+^{o}\mathbf{b})$ is the equation of the hyperbolic paraboloid,

$\mathbf{r}=\mathbf{q}^{o}\mathbf{C}+\mathbf{q}^{c}\mathbf{C}(\mathbf{Q}^{o}\mathbf{a}+\mathbf{Q}^{o}\mathbf{b})$ is the equation of the elliptical cone,

where $\mathbf{q}$ is an auxiliary vector whose tensor is equal to that of $\mathbf{r}$, and $\mathbf{a}$ and $\mathbf{b}$ are the axes of the ellipse at a unit's distance from vertex. $\mathbf{r}=\mathbf{q}^{o}\mathbf{C}+\mathbf{q}^{k}\mathbf{C}(\mathbf{Q}^{o}\mathbf{a}+\mathbf{Q}^{o}\mathbf{b})$ is the equation of the elliptical cylinder. To get the normal to a central quadric, say the ellipsoid, one writes down the scalar product of the dyadic

*The derivation of this and the following equations is omitted for want of space.

$$\frac{\mathbf{A}\,\mathbf{A}'}{a^2}+\frac{\mathbf{B}\,\mathbf{B}'}{b^2}+\frac{\mathbf{C}\,\mathbf{C}'}{c^2}$$

and the radius vector $\mathbf{o}\mathbf{a}+\mathbf{o}\mathbf{b}+\mathbf{o}\mathbf{c}$ which gives $\frac{\mathbf{o}\mathbf{A}}{a}+\frac{\mathbf{o}\mathbf{B}}{b}+\frac{\mathbf{o}\mathbf{C}}{c}$. Having the normal it is easy to construct the equation of the tangent plane.

It may be remarked that if the preceding equations were written in one of the usual notations, as, for instance, that used by Gibbs, they would become very complicated, though the concepts represented, as we have seen are simple enough.

5. Neither the usual vector equations, those just given, nor the cartesian equations, possess exclusive merits over the others. Each in turn can be superior to the others for certain purposes. It is easier to construct the curves and surfaces from the equations just given than from the cartesian equations since one can use the trigonometric tables. The equations for the *plane* curves do not differ except in compactness and convenience of notation from the vector equations commonly given. But those for space of three dimensions are new forms. Writers on vector analysis use the ϕ function for quadrics, which introduces very compact expressions, but ones essentially scalar. The use of the above equations makes the treatment of solid space analogous to that of plane and connects vector analysis much more closely to cartesian analysis.

NOTE ON THE TOTIENT OF A NUMBER.

By G. A. MILLER.

Like the preceding paper, "On the Totitives of Different Orders," MONTHLY, Vol. XI, p. 129, this note aims to point out the advantages of employing the theory of groups in the proofs of some of the elementary theorems of number theory.

The number of natural numbers which do not exceed the positive integer m and are prime to m is denoted by $\phi(m)$ according to Gauss.* This function of m is known by various names. English, French, and German writers respectively employ the following names: totient of m, indicator of m, Euler's ϕ-function of m. If $1, d_1, d_2, \ldots\ldots, d_\lambda, m$ are all the divisors of m it is well known that

$$\phi(1)+\phi(d_1)+\phi(d_2)+\ldots\ldots+\phi(d_\lambda)+\phi(m)=m.$$

The main object of this note is to show the connection between this formula and the theory of cyclic groups in a very elementary manner. Several closely related questions in number theory are also treated from the standpoint of group theory.

*Gauss, *Disquisitiones Arithmeticae*, Art. 38.

If d is any divisor of m then there is one and only one subgroup of order d in the cyclic group (G) of order m. This subgroup is cyclic. Hence there are just $\phi(d)$ operators of order d in G. Moreover, the order of every operator of G is a divisor of m. The given formula therefore says nothing more than that *the sum of the numbers of the operators of the different possible orders is equal to the order of* G. That is, this formula exhibits a very elementary property of cyclic groups and requires no further proof if the given properties of cyclic groups are assumed to be known. We proceed to give a simple proof of these properties.

Since d is a divisor of m any generator (s) of G must contain a cyclic subgroup of order d. It cannot contain more than one such subgroup because the 1st, 2d, ..., mth powers of s contain only d operators whose orders divide d,* and these powers give all the operators of G. These statements prove that G contains one and only one subgroup of order d and that this subgroup is cyclic.

The given formula may clearly be regarded as a special case of the following evident theorem. *If g is the order of any finite group and if* $1, d_1, d_2, \ldots, d_\gamma$ *are the orders of all of its cyclic subgroups, then*

$$\phi(1)+\phi(d_1)+\phi(d_2)+\ldots\ldots+\phi(d_\gamma)=g.$$

Since every non-cyclic group contains more than one cyclic subgroup of the same order, it follows that the given $\gamma+1$ divisors of g cannot all be different unless the group is cyclic. In this special case we arrive at the preceding formula, as has been proved above.

The given properties of the cyclic group may be employed to obtain very elementary proofs of the formulas $\phi(mn)=\phi(m)\phi(n)$ whenever m and n are prime to each other, and

$$\phi(m)=m\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\ldots\left(1-\frac{1}{p_\rho}\right)$$

where $p_1, p_2, \ldots\ldots, p_\rho$ are all prime factors of m.

The former of these two formulas gives the number of operators of highest order in the cyclic group of order mn, m and n being relatively prime. This cyclic group contains just one cyclic subgroup of each of the orders m and n, and it is the direct product of these two cyclic subgroups since they have only the identity in common. To obtain an operator of order mn it is necessary to multiply an operator of order m from the first subgroup into an operator of order n from the second subgroup, and all such products are of order mn. Hence the number of operators of highest order in the entire group is the product of the numbers of the operators of highest order from the two subgroups. In other words, $\phi(mn)=\phi(m)\phi(n)$.

All the subgroups of a cyclic group of order p^a, p being a prime, are con-

*From $(s^x)^d=1$ it follows that $xd=km$, or $x=km/d$. Hence x has one of the values $m/d, 2m/d, \ldots\ldots dm/d \bmod m$.

tained in its subgroup of order $p^{\alpha-1}$. That is, $\phi(p^\alpha)=p^\alpha - p^{\alpha-1}=p^\alpha(1-1/p)$. If $m=p_1^{\alpha_1}p_2^{\alpha_2}\ldots\ldots p_\rho^{\alpha_\rho}$, it follows from this and the preceding formula that

$$\phi(m)=\phi(p_1^{\alpha_1})\phi(p_2^{\alpha_2})\ldots\ldots\phi(p_\rho^{\alpha_\rho})$$

$$=p_1^{\alpha_1}p_2^{\alpha_2}\ldots\ldots p_\rho^{\alpha_\rho}\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\ldots\left(1-\frac{1}{p_\rho}\right)$$

$$=m\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\ldots\left(1-\frac{1}{p_\rho}\right).$$

It may be observed that the formula $\phi(p^\alpha)=p^{\alpha-1}(p-1)$ may be regarded as a special case of the formula $\phi(m^r)=m^{r-1}\phi(m)$, since $\phi(p)$ is evidently equal to $p-1$.

AN APPLICATION OF STIRLING'S INTERPOLATION FORMULA

By FRANK ELMORE ROSS.

Let write the values of $\sin x$ for $x=0$, π, etc., and form the series of differences, as in the ordinary process of interpolation. The result is as follows, Δ_n denoting the nth difference:

x	$\sin x$	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	.....
0	0						
		+1					
$\frac{1}{2}\pi$	+1		−2				
		−1		+2			
π	0		0		0		
		−1		+2		−4	
$3\pi/2$	−1		+2		−4		.....
		+1		−2		+4	
0	0		0		0		.....
		+1		−2		+4	
$\frac{1}{2}\pi$	+1		−2		+4		
		−1		+2			
π	0		0				
		−1					
$3\pi/2$	−1						

This scheme extends infinitely in both directions, the difference Δ_n forming a divergent series. It is found however that a convergent series for the

function thus tabulated can be constructed from the above series of differences, by making use of Stirling's interpolation formula.

$$f(a+nw)=f(a)+n\Delta_1+\frac{n^2}{2!}\Delta_2+\frac{n(n^2-1)}{3!}\Delta_3+\frac{n^2(n^1-1)}{4!}\Delta_4$$

$$+\frac{n(n^2-1)(n^2-2^2)}{5!}\Delta_5 \ldots\ldots (1).$$

In this formula Δ_n stands for the differences which lie in the same horizontal line $f(a)$, in the case of the even differences, and in the case of the odd differences, for the mean of those differences which lie just above and below this line. If we apply (1) to the above scheme, we get at once

$$\sin x=x\left[1-\frac{2(x^2-1)}{3!}+\frac{2^2(x^2-1)(x^2-2^2)}{5!}-\frac{2^3(x^2-1)(x^2-2^2)(x^2-3^2)}{7!}+\ldots\right.$$

$$\left.\ldots\ldots+\frac{(-2)^n(x^2-1)\ldots\ldots(x^2-n^2)}{(2n+1)!}-\ldots\ldots\right]\ldots\ldots (2),$$

in which x is expressed in terms of π. This series is convergent for all values of x, since the ratio of two successive terms

$$\frac{x^2-n^2}{n(2n+1)}\doteq\frac{1}{2}\quad(n\doteq\infty).$$

While this series is not of any practical use for computing this function, it is nevertheless believed to be new. From it we can readily obtain a simple series for π, as follows:

Since $\frac{\sin x}{x}\doteq\frac{1}{2}\pi$, $(x\doteq 0)$, we get by putting $x=0$ in (2),

$$\tfrac{1}{2}\pi=1+\frac{2}{3!}+\frac{2^2 2^2}{5!}+\frac{2^3 2^2 2^3}{7!}+\ldots\ldots+\frac{2^n 2^2 3^2\ldots\ldots n^2}{(2n+1)!}\ldots\ldots, \qquad \ldots\ldots(3)$$

which is convergent, the ratio of convergence being the same as in (2). This series is also believed to be new. It is found to be quite convenient for arithmetical computation. By means of it the value of π correct to eight decimal places was computed in half an hour. There are however better series for this purpose.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

No. 212 was also solved by Jeannette Brooks. Two solutions of No. 213 were received from G. B. Hudson.

215. Proposed by EDWIN L. RICH, Lehigh University.

Solve (1).....$x/a+y/b+c/z=3$,
(2).....$x/a+b/y+z/c=3$,
(3).....$a/x+y/b+z/c=3$.

I. Solution by J. SCHEFFER, Hagerstown, Md.

Subtracting (3) from (1) and (3) from (2) the results may easily be put in the form

$$(z/c)^2-(x/a-a/x)(z/c)=1 \text{.....(4)},$$
$$(y/b)^2-(x/a-a/x)(y/b)=1 \text{.....(5)},$$

whence $z/c=x/a,\ -a/x$; $y/b=x/a,\ -a/x$.

By substituting the first pair of values in (1),

$$x=a,\ y=b,\ z=c, \text{ and } x=\tfrac{1}{2}a,\ y=\tfrac{1}{2}b,\ z=\tfrac{1}{2}c.$$

By substituting the second pair of values we obtain

$$x=3a,\ y=-b/3,\ z=3c;$$
$$x=-a/3,\ y=3b,\ z=3c,$$
$$x=3a,\ y=3b,\ z=-c/3.$$

II. Solution by L. S. SHIVELY, Mt. Morris, Ill., and ELMER SCHUYLER, Brooklyn High School, NewYork.

Let $x/a=A$, $y/b=B$, and $z/c=C$. Then the original equations become

$$A+B+1/C=3 \text{.....(1)},$$
$$A+1/B+C=3 \text{.....(2)},$$
$$1/A+B+C=3 \text{.....(3)}.$$

Subtracting (2) from (1) and $B-1/B=C-1/C$, hence $B=C,\ -1/C$.

It can similarly be shown that

$$A=B=C \text{.....(4)}, \text{ and that}$$
$$A=-1/B;\ B=-1/C;\ C=-1/A \text{.....(5)}.$$

From (4) and (1), $2A+1/A=3$. The roots of this quadratic are $\frac{1}{2}$ and 1.
$\therefore A=B=C=\frac{1}{2}$, and $x=\frac{1}{2}a,\ y=\frac{1}{2}b,\ z=\frac{1}{2}c$; $A=B=C=1$, and $x=a,\ y=b,\ z=c$.

From (5) and (1) it is seen that $A=3$, $B=-\frac{1}{3}$, $C=3$.

$\therefore x=3a$, $y=-\frac{1}{3}b$, $z=3C$.

In like manner, $x=3a$, $y=3b$, $z=-\frac{1}{3}c$, and $x=-\frac{1}{3}a$, $y=3b$, $z=3c$.

Also solved by M. E. Graber, Grace M. Bareis, E. L. Sherwood, Christian Hornung, F. P. Matz, G. B. M. Zerr, and the Proposer.

216. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

Express by radicals the roots of $x^6+ax^4+bx^3+\frac{1}{4}a^2x^2+\frac{1}{2}abx+c=0$.

I. Solution by E. L. SHERWOOD.

$x^6+ax^4+\frac{1}{4}a^2x^2+bx^3+\frac{1}{2}abx+c=0$, $(x^3+\frac{1}{2}ax)^2+b(x^3+\frac{1}{2}ax)+c=0$, whence we have, by solving the quadratic

$$x^3+\tfrac{1}{2}ax+\frac{2b-b^2+4c}{4}=0, \text{ or } x^3+\tfrac{1}{2}ax+\frac{2b+b^2-4c}{4}=0,$$

whence by Cardan's method, Burnside and Panton, p. 108,

$$x=\sqrt[3]{p}+\frac{-H}{\sqrt[3]{p}},\quad \omega\sqrt[3]{p}-\frac{H}{\omega\sqrt[3]{p}},\quad \omega^2\sqrt[3]{p}-\frac{H}{\omega^2\sqrt{p}}$$

where $p=\frac{1}{2}[\sqrt{(G^2+4H^3)}-G]$, and $G=\frac{2b-b^2+4c}{4}$ or $\frac{2b+b^2-4c}{4}$, $H=\frac{1}{6}a$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Write the equation as follows:

$$x^2(x^2+\tfrac{1}{2}a)^2+bx(x^2+\tfrac{1}{2}a)+c=0.$$

Let $x(x^2+\frac{1}{2}a)=x^3+\frac{1}{2}ax=y$. $\quad\therefore y^2+by+c=0$.

$$\therefore y=\frac{-b\pm\sqrt{(b^2-4c)}}{2}.$$

Let ω be an imaginary cube root of unity, and let m, n be the roots of $t^2-yt-a^3/216=0$.

$\therefore x=m+n$, $x=\omega m+\omega^2 n$, $x=\omega^2 m+\omega n$.

As y has two values, the six values of x are expressed as radicals.

Also solved by J. Scheffer, G. W. Greenwood, Elmer Schuyler, F. P. Matz, and the Proposer.

217. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

Find the condition that $E\equiv x^5-bx^3+cx^2+dx-e$ shall be the product of a complete square and a complete cube.

I. Solution by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

The factors must be of the form

$$(x^2-2ax+a^2)(x^3+2ax^2+\frac{4a^2x}{3}+\frac{8a^3}{27}),$$

hence $b=\frac{5}{3}a^2$, $c=-\frac{10}{27}a^3$, $d=\frac{20}{27}a^4$, $e=\frac{8}{27}a^5$, and therefore

$$\sqrt[3]{\frac{3b}{5}}=-3\sqrt[3]{\frac{c}{10}}=\sqrt[4]{\frac{27d}{20}}=\sqrt[5]{\frac{27e}{8}}.$$

II. Solution by E. L. RICH, Lehigh University.

Consider the cube, $x^3+3px^2+3p^2x+x^3$, and the square, $x^2+2qx+q^2$. Their product is,

$$x^5+(2q+3p)x^4+(q^2+3p^2+6pq)x^3+(3pq^2+6p^2q+p^3)x^2+(2p^3q+3p^2q^2)x+p^3q^2.$$

Then the first condition is $2q+3p=0$ or $q=-\frac{3}{2}p$. The other conditions for the values of the coefficients gotten by equating coefficients, and substituting the first condition are,

$$b=\tfrac{15}{4}p^2,\quad c=-\tfrac{5}{4}p^3,\quad d=\tfrac{16}{4}p^4,\quad e=-\tfrac{9}{4}p^5.$$

Also solved by J. Scheffer, G. B. M. Zerr, Elmer Schuyler, and F. P. Matz.

218. Proposed by SAUL EPSTEEN, The University of Chicago, Chicago, Ill.

Prove that $\sum_{r=0}^{n}\frac{c_r}{r+1}=\frac{2^{n+1}-1}{n+1}$ where c_r is the coefficient of x^r in the expansion of $(1+x)^n$.

I. Solution by G. W. GREENWOOD, M. A. (Oxon), and HOWARD M. ARMSTRONG.

$$\sum_{r=0}^{r=n}\frac{c_r}{r+1}=\int_0^1[c_0+c_1x+c_2x^2+\ldots\ldots+c_nx^n]dx=\int_0^1(1+x)^ndx=\frac{2^{n+1}-1}{n+1}.$$

II. Solution by J. SCHEFFER, Kee Mar College.

By the binomial theorem

$$2^{n+1}-1=(1+1)^{n+1}-1=(n+1)+\frac{(n+1)n}{1.2}+\frac{(n+1)n(n-1)}{1.2.3}+\ldots\ldots$$

$$\therefore\ \frac{2^{n+1}-1}{n+1}=1+\frac{n}{1.2}+\frac{n(n-1)}{1.2.3}+\ldots\ldots=1+\tfrac{1}{2}\binom{n}{1}+\tfrac{1}{3}\cdot\binom{n}{2}+\ldots\ldots=\sum_{r=0}^{n}\frac{c_r}{n+1}.$$

III. Solution by CHRISTIAN HORNUNG, Heidelberg University, Tiffin, Ohio.

$$\sum_{r=0}^{n}\frac{c_r}{r+1}=c_0+\tfrac{1}{2}c_1+\tfrac{1}{3}c_2+\tfrac{1}{4}c_3+\ldots\ldots\frac{1}{r+1}+c_r\ldots\ldots+\frac{1}{n+1}c_n$$

$$=1=\tfrac{1}{2}n+\tfrac{1}{3}\cdot\frac{n(n-1)}{2!}+\tfrac{1}{4}\cdot\frac{n(n-1)(n-2)}{3!}+\ldots\ldots\frac{1}{n+1}$$

$$=\frac{1}{n+1}(n+1+\tfrac{1}{2}n(n+1)+\tfrac{1}{3}\cdot\frac{n(n-1)(n+1)}{2!}+\ldots\ldots+1)$$

$$= \frac{1}{n+1}(1+n+1+\frac{n(n+1)}{2!}+\frac{n(n+1)(n-1)}{3!}+\ldots\ldots+1)-1$$

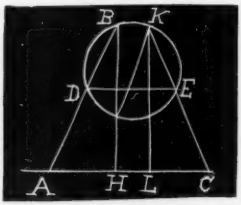
$$= \frac{1}{n+1}(1+1)^{n+1}-1 = \frac{2^{n+1}-1}{n+1}.$$

Demonstrated by M. E. Graber by the method of mathematical induction, and by Elmer Schuyler by means of the integral calculus. Also solved by G. B. M. Zerr.

GEOMETRY.

236. Proposed by J. R. HITT, San Marcos, Texas.

If two sides of a triangle pass through fixed points, the third side touches a fixed circle.



I. Solution by the PROPOSER.

Let the sides AB, BC of triangle ABC pass through the points D, E, respectively. Draw BH perpendicular to AC, from F, intersection of circumcircle of triangle DBE with BH, draw diameter FK. Draw BK, KL perpendicular to AC. Arc DBE is locus of B. Since $\angle ABH$ is constant, F is a fixed point. $\therefore K$ is fixed. FBK is a right angle. $\therefore BHLK$ is a rectangle. $\therefore KL=BH$, a constant. Hence AC touches the circle with center K and radius KL.

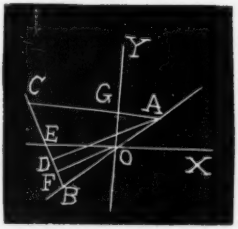
II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let O, G be the two fixed points; a, b, c the sides of the triangle; $OG=h$, $OA=r$, $\angle AOX=\angle FOB=\theta$, $AD=p$. Then $AD:OE=c:c-r$.

$$\therefore OE=\frac{p}{c}(c-r).$$

Also $\cos EOB=p/c$, or $EOB=\cos^{-1}(p/c)=\beta$.

$\therefore \rho\cos(\phi+\beta-\pi-\theta)=\frac{p}{c}(c-r)$ is the equation to CB. Let $\phi+\beta-\pi=\delta$.



$$\therefore \rho\cos\delta\cos\theta+\rho\sin\delta\sin\theta=\frac{p}{c}(c-r).$$

$$AG=\sqrt{(h^2+r^2-2rh\sin\theta)}.$$

$$\therefore h^2=h^2+r^2-rh\sin\theta+r^2-2r\sqrt{(h^2+r^2-2rh\sin\theta)}\cos A.$$

$$\therefore r^2\sin^2A-2rh\sin\theta\sin^2A=h^2\cos^2A-h^2\sin^2\theta.$$

$$\therefore \sin\theta=\frac{1}{h}[r\sin^2A\pm\cos A\sqrt{(h^2-r^2\sin^2A)}],$$

$$\cos\theta=\frac{\sin A}{h}[r\cos A\mp\sqrt{(h^2-r^2\sin^2A)}].$$

$$\therefore \frac{\rho r}{h}\sin A\cos(\delta-A)+\frac{pr}{c}-p=\pm\frac{\rho}{h}\sqrt{(h^2-r^2\sin^2A)}\sin(\delta-A).$$

Let $\frac{\rho}{h}\sin A\cos(\delta-A)+p/c=D$, $\frac{\rho}{h}\sin(\delta-A)=E$.

$\therefore rD-p=\pm E\sqrt{(h^2-r^2\sin^2 A)}$.

$(D^2+E^2\sin^2 A)r^2-2Drp=E^2h^2-p^2$.

$\therefore r=\dfrac{Dp}{D^2+E^2\sin^2 A}$, by differentiation.

$\therefore D^2h^2+E^2h^2\sin^2 A=p^2\sin^2 A$.

Substituting D and E and reducing $\rho^2c^2\sin^2 A+2\rho chp\sin A\cos(\delta-A)=p^2c^2\sin^2 A-p^2h^2$.

$\therefore \rho^2+2\rho H\cos(\delta-A)=G$ is the equation sought, and represents a circle.

Also solved by F. P. Matz.

243. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

What is the equation to the curve on which lie the centers of the inscribed circles in the right-angled triangles of hypotenuse h?

I. Solution by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

Take as axes the hypotenuse and its perpendicular bisector. Denoting the length of the hypotenuse by h, and one acute angle by a, the equations to the bisectors of the acute angles may be written

$$y=(x+\tfrac{1}{2}h)\tan\tfrac{1}{2}a,$$
$$y=-(x-\tfrac{1}{2}h)\tan(\tfrac{1}{4}\pi-\tfrac{1}{2}a),$$
$$=-(x-\tfrac{1}{2}h)\frac{1-\tan\frac{1}{2}a}{1+\tan\frac{1}{2}a}.$$

By eliminating a between these equations we get as the locus of their intersection, that is, of the center of the inscribed circle, $4(x^2+hy+y^2)=h^2$.

II. Solution by L. S. SHIVELY, Mt. Morris College, Mt. Morris, Ill.

It is evident that the angle subtended by the hypotenuse and with vertex at the center of the inscribed circle equals 135°. Since it is constant and equal to 135°, its vertex, and hence the center of the circle, lies upon that arc of a circle constructed upon h as a chord, to contain an angle of 135°. Construct such a circle and join O, its center, with A and B, the ends of h. Then $\angle AOB=90°$. Also $OA=\frac{1}{2}h\sqrt{2}$. Referred to O as origin of coördinates, the equation of the circle is $2x^2+2y^2=h^2$.

Also solved by J. Scheffer, E. L. Sherwood, G. B. M. Zerr, and Elmer Schuyler.

244. Proposed by O. W. ANTHONY, Head of the Mathematical Department, DeWitt Clinton High School, New York.

Upon the sides of a triangle as bases isosceles triangles with base angles of 30° are constructed. Show that the lines joining the vertices of these isosceles triangles form an equilateral triangle.

I. Solution by H. M. ARMSTRONG, Cooch's Bridge, Delaware.

Let the sides of the given triangle be denoted by the vectors 2α, 2β, and $2(\alpha+\beta)$. Let i denote a unit vector perpendicular to the plane of the triangle. Then

$$\rho_1=\alpha+\frac{1}{\sqrt{3}}i\alpha \quad\ldots\ldots(1), \qquad \rho_2=2\alpha+\beta+\frac{1}{\sqrt{3}}ei\beta \quad\ldots\ldots(2), \qquad \rho_3=\alpha+\beta-ei(\alpha+\beta) \quad\ldots\ldots(3)$$

are the vectors drawn to the vertices of the three isosceles triangles.

Hence the vectors of the sides of the new triangle are obviously $\rho_2-\rho_1$, $\rho_3-\rho_2$, $\rho_1-\rho_2$. By means of (1), (2), and (3), we find that $(\rho_2-\rho_1)^2=(\rho_3-\rho_2)^2=(\rho_1-\rho_2)^2$.

Hence $T(\rho_2-\rho_1)=T(\rho_3-\rho_2)=T(\rho_1-\rho_2)$. Therefore the triangle is equilateral.

II. Solution by G. W. GREENWOOD, M. A. (Oxon). Lebanon, Ill.

Denote the vertices by A, B, C, and the remaining vertices of the triangles adjacent to BC, CA, AB, by A', B', C', respectively. I will assume that the triangles are all exterior to the given triangle. Then

$$\begin{aligned}
&A'C=a/\sqrt{3}, \quad B'C=b/\sqrt{3}, \quad \angle A'CB'=60^\circ+C.\\
&A'B'^2=\tfrac{1}{3}[a^2+b^2-2ab\cos(60+C)]\\
&\qquad=\tfrac{1}{3}[a^2+b^2-ab(\cos C-\sqrt{3}\sin C)]\\
&\qquad=\tfrac{1}{6}[a^2+b^2+c^2+2\sqrt{3}\,ab\sin C]\\
&\qquad=\tfrac{1}{6}[a^2+b^2+c^2+\sqrt{3}\,abc/r],
\end{aligned}$$

where r is the radius of the circumcircle of the triangle ABC. The result shows that $A'B'=B'C'=C'A'$. The same result will follow if A' and A are on the same side of BC, etc.

Also solved by M. E. Graber, J. Scheffer, and G. B. M. Zerr. G. I. Hopkins refers to solution of Problem 96, Vol. 5, No. 4.

245. Proposed by J. H. M. MACLAGAN-WEDDERBURN, M. A., Chicago, Ill.

Given two lines of fixed length, $AB=a$, $BC=b$, perpendicular to each other. A line CP is drawn making $\angle BPC=\theta$. AD is the perpendicular to AB meeting CP in D. Find by Euclidean construction the angle θ such that $AD^2\cos^2\theta+b^2\sin^2\theta$ is a minimum.

Solution by the PROPOSER.

Draw any line CP. From B drop the perpendicular BL upon CP; draw AM parallel to PC meeting BL in M. Join CM. Then $CM^2=CL^2+LM^2=b^2\sin^2\theta+AD^2\cos^2\theta$. The locus of M is a circle described on AB as a diameter with O, the middle point of AB, as center. In order that CM^2, and therefore CM shall be a minimum, the line CM must pass through O.

The construction is therefore the following: Bisect AB in O; produce BOA to P so that $OP=OC$. Then BPC=required angle θ. [Newton's *Principia*, Vol. 2, Section 7, Proposition XXXIV, Scholium.]

245a. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

PCP', DCD' are conjugate diameters of an ellipse; PN, DM are the ordinates to the major axis at P and D; show $CM/PN=CN/DM=AC/BC$, and that AP and BD' are parallel, and that AP' is parallel to BD.

I. Solution by J. SCHEFFER, Kee Mar College, Hagerstown, Md.

Denote AC by a, BC by b, CP by a', CD by b', $\angle PCA$ by ϕ, and $\angle DCM$ by λ. From the well known relations of conjugate diameters of an ellipse, we have

$$\sin^2\phi=\frac{b^2(a^2-a_1^2)}{a_1^2(a^2-b^2)},\quad \cos^2\phi=\frac{a^2(a_1^2-b^2)}{a_1^2(a^2-b^2)},$$

$$\sin^2\lambda=\frac{b^2(a^2-b_1^2)}{b_1^2(a^2-b^2)},\quad \cos^2\lambda=\frac{a^2(b_1^2-b^2)}{b_1^2(a^2-b^2)}.$$

$$\therefore CM=b_1\cos\lambda=a\sqrt{\frac{b_1^2-b^2}{a^2-b^2}},\quad CN=a_1\cos\phi=a\sqrt{\frac{a_1^2-b^2}{a^2-b^2}},$$

$$PN=a_1\sin\phi=b\sqrt{\frac{a^2-a_1^2}{a^2-b^2}},\quad DM=b_1\sin\lambda=b\sqrt{\frac{a^2-b_1^2}{a^2-b^2}}.$$

$$\therefore \frac{CM}{PN}=\frac{a}{b}\sqrt{\frac{b_1^2-b^2}{a^2-a_1^2}}=\frac{a}{b},\ \text{since } a_1^2+b_1^2=a^2+b^2,\ \text{and}$$

$$\frac{CN}{DM}=\frac{a}{b}\sqrt{\frac{a_1^2-b^2}{a^2-b_1^2}}=\frac{a}{b}.$$

The coördinates of the point P are $\left(a\sqrt{\frac{a_1^2-b^2}{a^2-b^2}},\ b\sqrt{\frac{a^2-a_1^2}{a^2-b^2}}\right)$, and of A, $(a, 0)$; those of B are $(0, b)$, and of $D'\left(a\sqrt{\frac{b_1^2-b^2}{a^2-b^2}},\ -b\sqrt{\frac{a^2-b_1^2}{a^2-b^2}}\right)$; therefore

$$\text{slope of } AP=b\sqrt{\frac{a^2-a_1^2}{a^2-b^2}}\div a\left(\sqrt{\frac{a_1^2-b^2}{a^2-b^2}}-1\right)=-\frac{b}{a}\cdot\frac{\sqrt{(a_1^2-b^2)}+\sqrt{(a^2-b^2)}}{\sqrt{(a^2-a_1^2)}}.$$

$$\text{Slope of } BD'=-\frac{b}{a}\cdot\frac{1+\sqrt{\left(\frac{a^2-b_1^2}{a^2-b^2}\right)}}{\sqrt{\left(\frac{b_1^2-b^2}{a^2-b^2}\right)}}=-\frac{b}{a}\cdot\frac{\sqrt{(a^2-b_1^2)}+\sqrt{(a^2-b^2)}}{\sqrt{(b_1^2-b)}}.$$

By the relation $a_1^2+b_1^2=a^2+b^2$, both slopes are equal; therefore PA is parallel to BD'. In the same way, AP' is parallel to BD.

II. Solution by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

Since P and D are extremities of conjugate diameters, their coördinates may be written $(a\cos\phi, b\sin\phi)$, $(a\sin\phi, -b\cos\phi)$ where a and b are the semi-axes. We obtain the required result at once by disregarding the sign and considering only the absolute magnitudes of the coördinates. Equations to AP and BD are

$$\frac{x-a\cos\phi}{a-a\cos\phi}=\frac{y-b\sin\phi}{-b\sin\phi}, \text{ and } \frac{x-a\sin\phi}{-a\sin\phi}=\frac{y+b\cos\phi}{b+b\cos\phi},$$

and the inclinations are therefore

$$-\frac{b\sin\phi}{a(1-\cos\phi)} \text{ and } -\frac{b(1+\cos\phi)}{a\sin\phi},$$

which are easily seen to be equal.

Also solved by G. B. M. Zerr.

CALCULUS.

187. Proposed by L. T. JACKSON, St. Louis, Mo.

Find the area of the ellipse

$$x=a_1+a_2\cos\theta+a_3\sin\theta,$$
$$y=b_1+b_2\cos\theta+b_3\sin\theta.$$

I. Solution by H. B. LEONARD, B. S., Chicago, Ill.

Let $x'=x-a_1$, $y'=y-b_1$; then eliminating $\cos\theta$ and $\sin\theta$ in turn we obtain

$$b_2x'-a_2y'=\triangle\sin\theta,\ b_3x'-a_3y'=-\triangle\cos\theta,$$

where $\triangle=b_2a_3-a_2b_3$. The substitution $x''=b_2x'-a_2y'$, $y''=b_3x'-a_3y'$, which multiplies areas by $\triangle$, transforms the ellipse into the circle $x''=\triangle\sin\theta$, $y''=-\triangle\cos\theta$; *i. e.* $x''^2+y''^2=\triangle^2$.

The area of circle being $\pi\triangle^2$, the area of ellipse is $\pi\triangle=\pi(b_2a_3-a_2b_3)$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Transform the origin to the center by writing x for $x-a_1$, y for $y-b_1$.

$\therefore x=a_2\cos\theta+a_3\sin\theta$, $y=b_2\cos\theta+b_3\sin\theta$.

$$\text{Area}=\int ydx=\int_0^{2\pi}(b_2\cos\theta+b_3\sin\theta)(a_3\cos\theta-a_2\sin\theta)d\theta$$

$$=\int_0^{2\pi}(a_3b_2\cos^2\theta-a_2b_3\sin^2\theta-a_2b_2\sin\theta\cos\theta\pm a_3b_3\sin\theta\cos\theta)d\theta$$

$$=\pi(a_3b_2-a_2b_3).$$

Also solved by F. P. Matz, and the Proposer.

188. Proposed by SAUL EPSTEEN.

Evaluate $I=\int \frac{y(y^2+2)dy}{(y^4+3y^2+3)\sqrt{(y^2+1)}}$.

I. Solution by W. W. BEMAN, M. A., Professor of Mathematics in the University of Michigan.

Putting $y^2+1=z^2$ and $z-1/z=u$, the integrand becomes $\frac{1}{u^2+3}$; hence integral $=\frac{1}{3^{\frac{1}{2}}}\tan^{-1}\frac{1}{3^{\frac{1}{2}}}[(1+y^2)^{\frac{1}{2}}-(1+y^2)^{-\frac{1}{2}}]$.

II. Solution by CHRISTIAN HORNUNG, and M. E. GRABER, Tiffin, Ohio.

Let $y^2+1=u^2$, then $I=\int\frac{(u^2+1)du}{u^4+u^2+1}=\frac{1}{2}\int\frac{du}{u^2+u+1}+\frac{1}{2}\int\frac{du}{u^2-u+1}$

$$=\frac{1}{3^{\frac{1}{2}}}\left[\tan^{-1}\frac{2(y^2+1)^{\frac{1}{2}}+1}{3^{\frac{1}{2}}}+\tan^{-1}\frac{2(y^2+1)^{\frac{1}{2}}-1}{3^{\frac{1}{2}}}\right]$$

*** Dr. G. B. M. Zerr obtains the same result and reduces it to the form

$$\frac{1}{\sqrt{3}}\left[\pi+\tan^{-1}-\frac{(3y^2+3)^{\frac{1}{2}}}{y^2}\right].$$

Also solved by J. Scheffer, E. L. Sherwood, H. M. Armstrong, and G. W. Greenwood.
Solved with same result by F. P. Matz.

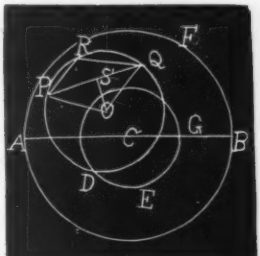
AVERAGE AND PROBABILITY.

130. Proposed by L. C. WALKER, A. M., Santa Barbara, Cal.

Three points are taken at random in a given circle, and a circle passed through them. The probability that the circle through the random points will be wholly in the given circle is $\frac{2}{5}$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let PRQ be the triangle formed by joining the three random points P, R, Q in the circle AFB; O the center of the circle circumscribing PRQ. Draw the circle DEG concentric with AFB, both with center C, so that $AD=GB=OP$. Since PQ is known in length and direction, and the angle PRQ is given, if OP is less than AC, the area of DEG represents the number of ways the three points can be taken, so that the circle circumscribing PRQ will lie wholly within AFB.



Let $PQ=2x$, $AC=r$, $\angle POS=\phi$, $\theta=\sin^{-1}(x/r)$, area of segment $PRQ=u$, ρ=angle PQ makes with some fixed line.

Then $PO=x\operatorname{cosec}\phi=r\sin\theta\operatorname{cosec}\phi$, $CD=r-x\operatorname{cosec}\phi=r(1-\sin\theta\operatorname{cosec}\phi)$.

Area $DEG = \pi r^2(1-\sin\theta\operatorname{cosec}\phi)^2$.

$u = x^2(\phi\operatorname{cosec}^2\phi - \cot\phi) = r^2\sin^2\theta(\phi\operatorname{cosec}^2\phi - \cot\phi)$.

An element of surface at Q is $4xdxd\rho = 4r^2\sin\theta\cos\theta d\theta d\rho$; at R it is $du = 2r^2\sin^2\theta\operatorname{cosec}^2\phi(1-\phi\cot\phi)d\phi$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of ϕ, θ and $\pi-\theta$; of ρ, 0 and 2π.

Hence, doubling as R may be on either side of PQ, we get for the required chance,

$$p = \frac{16\pi r^6}{\pi^3 r^6}\int_0^{\frac{1}{2}\pi}\int_\theta^{\pi-\theta}\int_0^{2\pi}\sin^3\theta\cos\theta\operatorname{cosec}^2\phi(1-\phi\cot\phi)(1-\sin\theta\operatorname{cosec}\phi)^2 d\theta d\phi d\rho$$

$$= \frac{32}{\pi}\int_0^{\frac{1}{2}\pi}\int_\theta^{\pi-\theta}\sin^3\theta\cos\theta\operatorname{cosec}^2\phi(1-\phi\cot\phi)(1-\sin\theta\operatorname{cosec}\phi)^2 d\theta d\phi$$

$$= \frac{2}{3\pi}\int_0^{\frac{1}{2}\pi}[2(\pi-2\theta)\sin 2\theta + 4 - 4\cos 4\theta - 3\sin^2 2\theta\cos 2\theta$$

$$+ 64\sin^4\theta\cos\theta\log\tan\tfrac{1}{2}\theta]d\theta = \frac{\pi}{6}.$$

Also solved by F. P. Matz.

Miscellaneous 146 was also solved by Jeannette Brooks.

PROBLEMS FOR SOLUTION.

ALGEBRA.

223. Proposed by THEODORE L. DELAND, Office of the Secretary of the Treasury, Washington, D.C.

An officer in the Treasury Department assigned three clerks to count a lot of silver dollars and when finished noted that there was an apparent difference in their efficiency; and, to determine the fact, gave to each a similar lot of the same amount to count, the only record made at the time being that A to count his lot alone, took three weeks longer, B took two weeks longer, and C took one week longer than it took for all working together to count the first lot. The best counter, on the record made, was given an efficiency mark of 93 on the scale of 100. What efficiency mark should, on the record, be given to each of the other two counters?

224. Proposed by G. W. GREENWOOD, M. A. (Oxon). Lebanon, Ill.

Show that, if none of the quantities x, y, z is zero, the result of eliminating them from

$$(x+y)(x+z) = bcyz \quad \ldots\ldots (1),$$
$$(y+z)(y+x) = cazx \quad \ldots\ldots (2),$$
$$(z+x)(z+y) = abxy \quad \ldots\ldots (3),$$

is $\begin{vmatrix} \pm a, & 1, & 1 \\ 1, & \pm b, & 1 \\ 1, & 1, & \pm c \end{vmatrix} = 0.$ [Oxford, 1896.]

225. Proposed by H. M. ARMSTRONG, Cooch's Bridge, Delaware.

If $\alpha=ax+cy+bz$ .....(1), $\beta=cx+by+az$ ..... (2), $\gamma=bx+ay+cz$ ..... (3), show that $\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma=(a^3+b^2+c^3-3abc)(x^3+y^3+z^3-3xyz)$.

GEOMETRY.

248. Proposed by CHRISTIAN HORNUNG, Heidelberg University, Tiffin, Ohio.

Given AB, BC in a straight line, to produce it to D so that $AD.CD=BD^2$.

249. Proposed by W. W. BEMAN, The University of Michigan.

Given the distances of a point in the plane of a square from three of its vertices, to find the side of the square.

250. Proposed by W. W. BEMAN, The University of Michigan.

Given the distances of a point in the plane of an equilateral triangle from the vertices; to find the side of triangle. [Perkins' *Geometry*, Olney's *Geometry*.]

CALCULUS.

189. Proposed by J. E. SANDERS, Hackney, Ohio.

Solve $d^2y/dx^2=-\beta^2(p+y)$, p and β being constants. The initial conditions are $y=0$ for $x=0$, l; $dy/dx=0$ for $x=l/2$. [Merriman's *Mechanics*, 9th Ed., 1903, §62.]

190. Proposed by SAUL EPSTEEN, The University of Chicago, Chicago, Ill.

$$\int_0^\infty \frac{\sin x\cos\beta x}{x}dx, \quad \int_0^\infty \frac{\sin\alpha x\cos x}{x}.$$

MECHANICS.

170. Proposed by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

Prove that the moment of inertia of an ogival head rotating about its geometrical axis is $\frac{\pi w}{g}\int_0^{R\sqrt{(4n-1)}} y^4dx$, where w is the weight in pounds of a cubic foot of material, R the radius of the base of the ogive, and n the diameter of projectile.

DIOPHANTINE ANALYSIS.

123. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Of two numbers $a_ib_ic_id_ie_i$ $(i=1, 2)$ it is given that their 10 digits a_1, ..., e_2 form a permutation of 0, 1, ..., 9, and that the sum of the two is $x3951$. Give an immediate evaluation of x; also list the possible pairs a_1, a_2; ....; e_1, e_2.

AVERAGE AND PROBABILITY.

157. Proposed by J. E. SANDERS, Hackney, Ohio.

A box contains n tickets numbered from 1 to n. How many draws, on the average, will it take to draw all the numbers, each ticket being replaced before drawing again? What is the numerical result for $n=2$ and $n=6$?

NOTE.—Problems and solutions in the departments of Geometry, Calculus, Mechanics, and Average and Probability should be sent to B. F. Finkel; and those in the departments of Algebra, Diophantine Analysis, Miscellaneous, and Group Theory should be sent to Dr. Saul Epsteen. Our contributors should carefully observe this notice if proper credit for contributions is to be given.

NOTES.

The University of Missouri has come into possession of the very excellent Loan Exhibit of Mathematical Models which were exhibited at the World's Fair at St. Louis, by Schilling of Halle, Germany. F.

Dr. G. B. Halsted has been made Foreign Associate and Honorary Professor of Mathematics in the University of Tempio, Italy; and also a Fellow of the Royal Astronomical Society. F.

At the last session of the Missouri State Teachers' Association, the mathematics section of that organization resolved to organize into a society to be known as the Missouri Society of Teachers of Mathematics, the aim of the society being the improving of methods and subject-matter of mathematical instruction in the schools of the State, the distributing of literature pertaining to mathematical instruction, and the promoting of social relations between teachers in the secondary schools, graded schools, and colleges of the State. Dr. Hedrick was chosen chairman of the committee on organization, and Dr. Ames chairman of the Society. F.

Professor P. Barbarin, President of the Societé des Sciences Physiques et Naturelles de Bordeaux has acquired the right to translate into French Dr. Halsted's *Rational Geometry* of which he has a long review in the current number of *l'Enseignement Mathematique;* and Yoshio Mikami will translate into Japanese and publish in Japan Dr. Halsted's Vice-Presidential Address to the Association for the Advancement of Science, the subject of which is, The Message of Non-Euclidean Geometry. F.

BOOKS.

The Principles of Mathematical Physics. By Henri Poincairé. Translated by George Bruce Halsted. Pamphlet, 24 pages.

This is a translation of the French savant's address delivered before the International Congress of Arts and Science, at St. Louis, Mo., September, 1904. B. F. F.

Principles of Physics, for Academies, High Schools, and Normal Schools. By Louis Begeman, M. S., Professor of Physics, Iowa State Normal School. 8vo. Cloth, 267 pages. Chicago: Ainsworth & Co.

A splendid work, well suited to elementary instruction. B. F. F.

Exercises in Algebra. By Edward R. Robbins and F. H. Summerville, William Penn Charter School, Philadelphia, Penn. 8vo. Half Leather, 173 pages. New York and Chicago: American Book Co.

This little book will be found helpful to those teachers wishing to supplement the exercises in the text-book they are using. B. F. F.

An Elementary Treatise on the Differential Calculus, founded on the Method of Rates. By William Woolsey Johnson, Professor of Mathematics at the U. S. Naval Academy, Annapolis, Md. Small 8vo. Cloth, xiv+406 pages, 70 figures. Price, $3.00. New York: John Wiley & Sons.

This book is not a revision of the earlier work published by the same author in connection with J. M. Rice, but it is essentially new. The order of subjects differs from the older work, though the method of treatment is very much the same. The author still employs the Newtonian conception of rates, but more closely connected with the method of limits. Each chapter has at the close a fine collection of problems demanding in a variety of ways the applications of the principles therein established. B. F. F.

Statistical Methods with Special Reference to Biological Variation. By C. B. Davenport, Head of the Department of Experimental Biology and Director of Station for Experimental Evolution of the Carnegie Institution. Second Edition, Enlarged. 16mo. Morroco, viii+223 pages. Price, $1.50. New York: John Wiley & Sons.

This edition in addition to the matter contained in the first edition, embodies many of the new statistical methods elaborated chiefly by Prof. Karl Pearsons, his students and associates, and presents a summary of the results gained by these methods. B. F. F.

A Geometrical Political Economy. Being an Elementary Treatise on the Method of Explaining some of the Theories of Pure Economic Science by means of Diagrams. By H. Cunynghame, C. B., M. A., Oxford. At the Clarendon Press, 1904, 128 pp.

This little book is addressed to economists rather than to mathematicians. According to the author (p. 127) "The chief function of Mathematics as applied to Economics is, not to solve problems, but to help us to comprehend truths, which when we have comprehended we may discard the Mathematics as we take down a scaffolding when the building is finished.

Chapter I contains a bibliography of books on this mode of treating Political Economy, supplementing the one in Jevon's *Theory of Political Economy* (3rd ed. 1879).

Chapter II begins with a defense of mathematics against the prejudice which many

students of economics feel. "Mathematical reasoning," says the author, "is only 'ordinary reasoning' assisted by a shorthand mode of expression that enables a proposition to be put in a line and visible in one glance of the eye rather than spread over ten or twelve pages of print. The remainder of the chapter discusses, as the title indicates, Geometrical Diagrams. The fundamental notions of drawing to a scale and change of axes are introduced by means of simple examples, then the terms: axes of coördinates, abscissa, origin, are defined and the methods of adding, subtracting, multiplying and dividing curves are explained and illustrated. Curves are divided into two classes: those which group and present facts, and those which represent laws; the latter are the ones dealt with in this treatise.

In Chapter III, on Demand Curves, money is taken as a standard of value, the reason being that the author desires to have for the simplicity of exposition a unit which may be regarded as invariable. Otherwise, if we consider the demand for (say) linen in terms of cloth, a diminishing utility of linen as it is supplied, involves a changing utility of cloth which its supply brings about (p. 105). In Chapter IV it is shown that a demand curve is a curve of final-utility prices; and one sees easily the characteristic feature of demand curves: that they are *descending curves, i. e.* with increasing quantity x, the price y decreases. The product xy denotes the total amount paid for the quantity x, and since people would not give a larger total sum for a smaller quantity of an article than for a larger, the demand curve will cut a rectangular hyperbola xy=constant, negatively (p. 42).

In Chapter V it is shown that Supply Curves are of three types: ascending (agricultural form), constant (industrial form), descending (manufacturing form). The intersection of the demand and supply curves gives the quantity of the article produced and sold.

Chapters VI—X are devoted chiefly to applications of the principles and methods explained above.

In Chapter XI the author explains the "Marshall's Curves," the applications which are made however go beyond those of the inventor. These curves were introduced for the purpose of eliminating the money standard in comparing two articles. Instead of having abscissa and ordinate denote quantity and prices per article, Marshall makes them denote total quantities exchanged. Thus if x and y denote abscissa and ordinate of a point on the price curve, where x=quantity taken, y=price per article, the abscissa and ordinate on a Marshall curve would x and $y=xy$.

On page 83, Fig. 41, two different points are denoted by the letter S. This should be changed to obviate confusion. On page 2 curves are spoken of as those which do and those which do not obey some law. By this the author seems to mean curves for which equations are or are not known. On page 7, it is stated that "mathematics is the science of quantity." In this connection Professor Bocher of Harvard University says:* "the old idea that mathematics is the science of quantity . . . . has pretty well passed away among those mathematicians who have given any thought to the question of what mathematics really is."

On the whole Mr. Cunynghame has carried out creditably his delicate task of writing a book on a subject which is neither Mathematics nor Economics, but is, as the title page states, "an elementary treatise on the method of explaining some of the theories of pure economics by means of diagrams."

SAUL EPSTEEN.

**The Fundamental Conceptions and Methods of Mathematics.* By Maxime Bocher. Address delivered before the Department of Mathematics of the International Congress of Arts and Science, St. Louis, September 20, 1904. Printed in the Bulletin of the American Mathematical Society, Vol. XI, No. 3, Dec. 1904, pp. 115—135.

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XII. MARCH, 1905. No. 3.

A GENERAL THEOREM IN LOCAL PROBABILITY.

By Professor ROBERT E. MORITZ, University of Washington, Seattle.

The theorem which forms the objective point of this paper deals with the probability that of n points, distributed at random on a line of length a, the distance between every two shall exceed an arbitrarily assumed distance b. The proof was suggested by the following problem given by Czuber.*

Problem: *In a straight line $AB=a$ two points are assumed at random. It is required to find the probability that their distance apart shall exceed a given length b.*

Czuber gives two solutions which are essentially as follows:

First Solution. The probability that the first point lies within an interval δx_1 is $\delta x_1/a$. The probability that the second point lies within an interval δx_2 is $\delta x_2/a$. The probability that the two points lie simultaneously within these respective intervals is $\delta x_1 \delta x_2/a^2$.

Let each interval be diminished indefinitely, the limiting value $dx_1 dx_2/a^2$ represents the probability that two points, placed at random on the line AB, are situated at the respective distances x_1 and x_2 from A.

All positions on the line are mutually exclusive and equally possible. The positions favorable to the conditions of our problem are subject to the restriction that the difference between x_1 and $x_2 > b$.

Suppose $x_1 < x_2$, and let x_1 be taken at random. Then every value of x_2 between the limits x_1+b and a gives a favorable case, while x_1 may have any value from 0 to $a-b$. The probability of the favorable cases subject to the condition $x_1 < x_2$ is therefore given by the double integral

$$\int_0^{a-b}\int_{x_1+b}^{a} \frac{dx_1 dx_2}{a_2} = \frac{1}{a^2}\int_0^{a-b}(a-b-x_1)dx_1 = \tfrac{1}{2}\left(\frac{a-b}{a}\right)^2$$

*Geometrische Wahrscheinlichkeiten, u. Mittelwerte, §11.